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LETTER TO THE EDITOR

The q-deformation of symmetric functions and the symmetric group

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Abstract. The q-deformation of symmetric functions is introduced leading to q-analogues of many well known relationships in the theory of symmetric functions. q-deformed scalar products are developed and used to define q-dependent symmetric functions. The symmetric functions commonly associated with the names Hall-Littlewood, Schur and Jack are all special cases of the q-deformation of Macdonald's new symmetric functions $P_\lambda(s, t)$. A q-analogue of the spin and ordinary characters of S_n is given and illustrated by the explicit calculation of examples of q-deformed characters. The methods used are closely parallel to those of quantum groups.

Symmetric functions find many applications in physics. The Schur functions (or S-functions) arise naturally in the character theory of the symmetric group while the Hall-Littlewood functions find application in the theory of solitons (Nimmo 1990). Macdonald has given a definitive account of both of these types of symmetric functions (Macdonald 1979) as well as their various specializations such as Schur's Q-functions that arise in the projective representations of the symmetric group. Throughout we shall follow the notation defined in Macdonald's book (Macdonald 1979).

The characters of the Hecke algebra $H_n(q)$ of type A_{n-1} may be calculated by generalizing the power sum symmetric functions as (King and Wybourne 1990)

$$p_r(q; x) = \sum_{a,b=0,a+b+1=r}^{r-1} (-1)^b q^a s_{(a+1,1^b)}(x) \tag{1}$$

and for $\rho = (\rho_1, \rho_2, \dots)$ letting

$$p_\rho(q; x) = p_{\rho_1}(q; x) p_{\rho_2}(q; x) \dots . \tag{2}$$

This could be thought of as a particular type of q-deformation of the power sum symmetric function. When $q \to 1$ (1) becomes the standard relationship between power sum symmetric functions $p_r(x)$ and Schur functions $s_\lambda(x)$. Indeed if we introduce a q-number $[n]_q$ such that

$$[n]_q = 1 + q + q^2 + \dots + q^n \tag{3}$$

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with $[0]_q = 1$, we may obtain a q-analogue of the usual relationship between the power sum symmetric functions $p_n(x)$ and, for example, the complete symmetric functions $h_n(x)$ by writing

$$[n-1]_q!\,h_n = \begin{vmatrix} p_1 & -[0] & & & \\ p_2 & p_1 & -[1] & & \\ p_3 & p_2 & p_1 & -[2] & \\ \vdots & \vdots & & \ddots & \\ p_n & p_{n-1} & \cdots & & p_1 \end{vmatrix} \tag{4}$$

where the power sum symmetric functions $p_r = p_r(q; x)$. We have, for example

$$h_3(x) = \frac{p_{1^3} + (q+2)p_{21} + (q+1)p_3}{(1+q)(1+q+q^2)}. \tag{5}$$

For $q = 1$ we obtain the usual result

$$h_3 = \frac{p_{1^3} + 3p_{21} + 2p_3}{6}. \tag{6}$$

The coefficients in (6) are just the numbers

$$c(\lambda) = z_\lambda^{-1} \tag{7}$$

where as usual (Macdonald 1979, p 17)

$$z_\lambda = \prod_i i^{m_i} m_i! \tag{8}$$

that is

$$h_n(x) = \sum_{|\lambda|=n} z_\lambda^{-1} p_\lambda(x). \tag{9}$$

The q-analogue of (9) is

$$h_n(x) = \sum_{|\lambda|=n} (z_\lambda^q)^{-1} p_\lambda(q; x) \tag{10}$$

where

$$z_\lambda^q = \prod_i [i-1]_q^{m_i} [m_i - 1]_q!\,. \tag{11}$$

The above results suggest that it could be useful to explore other types of q-deformations of symmetric functions not unlike those considered under the term of quantum groups (cf Drinfeld 1988). It is possible to consistently define various types of q-deformations of symmetric functions such as in terms of the q-numbers

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}. \tag{12}$$

Macdonald has introduced a more general symmetric function $P_\lambda(s, t)$ (Macdonald 1988) indexed by partitions λ and involving two indeterminates s and t. It is useful to consider their q-deformation.

Let $x=(x_1, x_2, \ldots)$ and $y=(y_1, y_2, \ldots)$ be two sequences of indeterminates over $\mathcal{Q}(t)$ where t is another indeterminate and define (Macdonald 1979)

$$P(x, y; t)=\prod_{i,j}\frac{(1-tx_iy_j)}{(1-x_iy_j)}. \tag{13}$$

A scalar product $\langle\,,\rangle_{(t)}$ over $\mathcal{Q}(t)$ may be defined as

$$\langle p_\lambda, p_\mu\rangle_{(t)}=\delta_{\lambda\mu}z_\lambda(t) \tag{14}$$

where

$$z_\lambda(t)=z_\lambda\prod_i(1-t^{\lambda_i})^{-1}. \tag{15}$$

Let s be another independent indeterminate and define a scalar product $\langle\,,\rangle_{(s,t)}$ over $\mathcal{Q}(s, t)$ as follows

$$\langle p_\lambda, p_\mu\rangle_{(s,t)}=\delta_{\lambda\mu}z_\lambda(s, t) \tag{16}$$

where

$$z_\lambda(s, t)=z_\lambda\prod_i\frac{(1-s^{\lambda_i})}{(1-t^{\lambda_i})}. \tag{17}$$

We can define a q-deformed scalar product $\langle\,,\rangle^{(q)}_{(s,t)}$ over $\mathcal{Q}_q(s, t)$ as

$$\langle p_\lambda, p_\mu\rangle^{(q)}_{(s,t)}=\delta_{\lambda\mu}z^q_\lambda(s, t) \tag{18}$$

where

$$z^q_\lambda(s, t)=\prod_i[i-1]^{m_i}_q[m_i-1]_q!\prod_j^{l(\lambda)}\frac{(1-s^{[\lambda_j-1]_q})}{(1-t^{[\lambda_j-1]_q})}. \tag{19}$$

We call $P^q_\lambda(s, t)$ the q-deformation of the symmetric function $P_\lambda(s, t)$.

A q-analogue of (13) can be defined as

$$\mathcal{P}_q=\prod_{i,j}\left\{\frac{(tx_iy_j; s)_\infty}{(x_iy_j; s)_\infty}\right\}_q \tag{20}$$

where

$$(a; s)_\infty=\prod_{r=0}^{\infty}(1-as^r) \tag{21}$$

and the notation $\{b\}_q$ is used for the q-deformed expansion of b. We then have

$$\mathcal{P}_q(x, y; s, t)=\sum_\lambda z^q_\lambda(s, t)^{-1}p_\lambda(x)p_\lambda(y). \tag{22}$$

The above result follows upon computing $\exp(\log\mathcal{P}_q)$ and leads to a general definition of q-deformed symmetric functions in which the usual symmetric functions, such as Hall–Littlewood, Jack, Schur and Q-functions, become associated with particular values (q, s, t). Setting $q=1$ yields the usual symmetric functions but values of q not a root of unity yield q-deformed symmetric functions. For example, $P^q_\lambda(0, t)$ or simply $P^q_\lambda(t)$ is the q-deformation of the Hall–Littlewood symmetric function. The scalar product $\langle\,,\rangle^{(q)}_{(t)}$ over $\mathcal{Q}(t)$ for this case follows immediately from specialization of (16) and (17).

Let us introduce another symmetric function $Q^q_\lambda(t)$ related to $P^q_\lambda(t)$ by a scalar $b^q_\lambda(t)$ as follows

$$Q^q_\lambda(t)=b^q_\lambda(t)P^q_\lambda(t) \tag{23}$$

where

$$b^q_\lambda(t)=\prod_{i\geq 1}\phi_{m_i(\lambda)}(t) \qquad \phi^q_n(t)=\prod_{j\geq 1}^{n}(1-t^{[j-1]_q}). \tag{24}$$

Then

$$\langle P^q_\lambda(t), Q^q_\mu(t)\rangle^{(q)}_{(t)}=\delta_{\lambda\mu} \tag{25}$$

i.e. $P^q_\lambda(t)$, $Q^q_\lambda(t)$ are dual bases of $\Lambda_{\mathcal{Q}}$ for the scalar product $\langle\,,\rangle^{(q)}_{(t)}$. It is easy to see that

$$\begin{aligned}Q^q_\lambda(t)&=\prod_{i<j}\left\{\frac{1-\delta_{ij}}{1-t\delta_{ij}}\right\}_q q^q_\lambda(t)\\&=\prod_{i<j}(1+(t-1)\delta_{ij}+(t^{1+q}+t)\delta^2_{ij}+\ldots)q^q_\lambda(t)\end{aligned} \tag{26}$$

where $q^q_\lambda(t)$ is the projection of $Q^q_\lambda(t)$, δ_{ij} is Young's raising operator whose action is defined as $\delta_{ij}q(\lambda_1,\ldots,\lambda_i,\ldots,\lambda_j,\ldots)=q(\lambda_1,\ldots,\lambda_i+1,\ldots,\lambda_j-1,\ldots)$.

We are now in a position to give a q-analogue of the characters of S_n. The characters χ^λ_μ are elements of a transition matrix that relates Schur functions to power sum symmetric functions via

$$s_\lambda=\sum_\mu z^{-1}_\mu\chi^\lambda_\mu p_\mu \tag{27}$$

and in the case of the spin characters ζ^λ_ν the Q-functions are related to the power sum symmetric functions via

$$Q_\lambda=2^{[(l(\lambda)+l(\nu)+1)/2]}\sum_\nu z^{-1}_\nu\chi^\lambda_\nu p_\nu. \tag{28}$$

For $s=t$, $P^q_\lambda(s,t)$ reduces to the q-deformed Schur function s^q_λ and for $s=0$ and $t=-1$, $P^q_\lambda(s,t)$ reduces to the q-deformed Q-function Q^q_λ. Thus we are led to the q-analogue of (27) and (28) as

$$s^q_\lambda=\sum_\mu(z^q_\mu)^{-1}\chi^\lambda_\mu(q)p^q_\mu \tag{29}$$

$$Q^q_\lambda=2^{[(l(\lambda)+l(\nu)+1)/2]}\sum_\nu(z^q_\nu)^{-1}\zeta^\lambda_\nu(q)p^q_\nu. \tag{30}$$

The characters $\chi^\mu_\lambda(q)$ may be calculated using the following algorithm.

1. Expand the S-function s^q_λ in terms of complete symmetric functions h^q_λ using the following

$$s^q_\lambda=\prod_{i<j}(1-\delta_{ij})h^q_\lambda \tag{31}$$

where $h^q_\lambda\equiv h^q_{\lambda_1}h^q_{\lambda_2}h^q_{\lambda_3}\ldots$.

2. Expand h^q_n in terms of power sum symmetric functions p^q_μ as follows

$$h^q_n=\sum_\mu(z^q_\mu)^{-1}p^q_\mu. \tag{32}$$

Table 1. q-dependent characters of S_4.

	1^4	21^2	22	31	4
4	1	1	1	1	1
31	$q+q^2+q^3$	q	-1	0	-1
22	q^2+q^4	$1-q$	$1+q$	-1	0
21^2	$q^3+q^4+q^5$	-1	$-q$	0	1
1^4	q^6	$-q$	q	1	-1

Table 2. q-Kostka–Foulkes polynomials $K^q_{\lambda\mu}(t)$ for $n=4$.

	4	31	22	21^2	1^4
4	1	$t^{[0]}$	$t^{[1]}$	$t^{[2]}$	$t^{[5]}$
31		1	$t^{[0]}$	$t^{[0]}+t^{[1]}$	$t^{[2]}+t^{[3]}+t^{[4]}$
22			1	$t^{[0]}$	$t^{[1]}+t^{[3]}$
21^2				1	$t^{[0]}+t^{[1]}+t^{[2]}$
1^4					1

3. $\chi^{\mu}_{\lambda}(q)$ is now found by comparison of the coefficient of p^q_μ on both sides of (29).

The q-dependent characters for S_4 following from (29) are given in table 1.

It is easy to see that for $q=1$ we get the usual characters of S_4.

As a consequence of the above discussion, we are able to give a q-analogue of Kostka–Foulkes polynomials $K_{\lambda\mu}(t)$ (cf Macdonald 1979, p 124) which appear in the following expansions

$$s^q_\lambda(x)=\sum_\mu K^q_{\lambda\mu}(t)P^q_\mu(x;t) \tag{33}$$

$$Q^q_\lambda(x;t)=\sum_\mu K^q_{\lambda\mu}(t)S^q_\mu(x;t). \tag{34}$$

As an example the q-Kostka–Foulkes polynomials $K^q_{\lambda\mu}(t)$ for $n=4$ are given in table 2.

In the preceding remarks we have noted that it is possible, and we consider fruitful, to study the q-deformation of symmetric functions in a rather similar manner to that used in quantum groups. We expect the study of the q-deformation of symmetric functions to provide the basis for a more unified treatment of the theory of symmetric functions and we hope eventually to lead to physical applications.

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References

Drinfeld V G 1988 *Sov. Math. Dokl.* **36** 212–6

King R C and Wybourne B G 1990 *J. Phys. A: Math. Gen.* **23** L1193–7

Macdonald I G 1979 *Symmetric Functions and Hall Polynomials* (Oxford: Oxford University Press)

—— 1988 *A New Class of Symmetric Functions, Publ. IRMA Strasbourg, Actes 20th Seminaire Lotharingien* **372/S-20** pp 131–71

Nimmo J J C 1990 *J. Phys. A: Math. Gen.* **23** 751–60